

PROJECTIVE RANKS OF COMPACT HERMITIAN SYMMETRIC SPACES

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Abstract

Let M be a compact irreducible Hermitian symmetric space and write $M = G/K$, with G the group of holomorphic isometries of M and K the stability group of the point of $0 \in M$. We determine the maximal dimension of a complex projective space embedded in M as a totally geodesic submanifold.

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Introduction

Let M be a simply connected compact complex manifold carrying an Hermitian metric of everywhere nonnegative holomorphic bisectional curvature. In [11] Mok proved that if the second Betti number $b_2(M) = 1$, then M is biholomorphic to an irreducible compact Hermitian symmetric space. It was earlier proved by Siu and Yau [14] and Mori [12] that when the above curvature is everywhere positive M is biholomorphic to a complex projective space. In each of the three papers cited above the existence of a minimally embedded projective line in M plays a crucial role.

It turns out that such minimally embedded projective lines are totally geodesic in M . In this paper we study the maximal totally geodesic complex submanifolds of M which are biholomorphic to a complex projective space. We call the dimension of such a submanifold the projective rank of M . We calculate the projective ranks of each of the irreducible compact Hermitian symmetric spaces. The results are given in Section 5. In Sections 1 through 4 we develop the techniques used to make these calculations and in the last section we discuss the degrees of the holomorphic totally geodesic maps $\varphi : \mathbb{P}_{\mathbb{C}}^r \rightarrow M$ where $r =$ projective rank of M . We also discuss the conjugacy of these maximal totally geodesic complex projective spaces in M under the action of the group of isometries of M .

We shall make use of the work of Chen and Nagano [5] on totally geodesic submanifolds of symmetric spaces. Several authors have studied the question of minimal or energy minimizing maps from S^2 to a compact symmetric space. The interested reader may consult [2] and [3] for connections with the present paper. It is the recent work of Robert Bryant [18] which motivates the author to revive these results which were summarized without proofs in the article [19]. Unlike Bryant, the approach taken here is very much an algebraic one with only an occasional nod to the topological and analytic methods which underlie many of the foundational results. We thank the referee of an earlier version of this paper for pointing out an error in our original discussion of the case of the quadrics.

1 Preliminaries on Hermitian Symmetric Spaces.

A complex manifold M_0 (noncompact) with Hermitian metric h is an Hermitian symmetric space or H.S.S. if each point of M_0 is an isolated fixed point of an involutive holomorphic isometry of M_0 . Let G_0 denote the connected component of the group of holomorphic isometries of M_0 . Then G_0 is a connected Lie group which acts transitively on M_0 . Fix $p_0 \in M_0$ and let $K \subset G_0$ be the isotropy group of p_0 so $M_0 = G_0/K$. We introduce the following notations:

\mathfrak{g}_0 : Lie algebra of G_0

s_0 : involutive isometry having p_0 as isolated fixed point

\tilde{k} : Lie algebra of K

$\mathfrak{g}_0 = k + m_0$: decomposition of \mathfrak{g}_0 into the $+1$ and -1 eigenspaces of $\sigma = \text{ad}(s_0)$

$\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$: complexification of \mathfrak{g}_0

$m = m_0^{\mathbb{C}}$: complexification of m_0

$\mathfrak{g}_c = k + m_c$: compact real form of \mathfrak{g} where $m_c = im_0$ (cf., [7; V 2.1]).

Let t be a Cartan subalgebra of k . Then t is a Cartan subalgebra of \mathfrak{g}_0 and \mathfrak{g}_c and $\mathfrak{h} = t^{\mathbb{C}}$ is a Cartan subalgebra of \mathfrak{g} . In fact, G_0 is a connected centerless semi-simple Lie group. The center $Z(K)$ of K is a torus whose dimension is the number of simple direct factors of G_0 and K is the centralizer in G_0 of $Z(K)$.

Let Φ be the \mathfrak{h} -root system of \mathfrak{g} so

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha \in \Phi} \mathfrak{g}_{\alpha}$$

Φ_K : compact roots; i.e., \mathfrak{h} -root system of $k^{\mathbb{C}}$

Φ_M : noncompact roots, i.e., $\{\alpha \in \Phi : \mathfrak{g}_{\alpha} \subseteq m\}$

z : central element of k such that $J = \text{ad}(z)$ induces the complex structure of M_0

m^{\pm} : $(\pm i)$ -eigenspace of J on m

$p : k^{\mathbb{C}} + m^{-}$, parabolic subspace of \mathfrak{g} that is the sum of nonnegative eigenspaces of $\text{ad}(iz)$:
 $\mathfrak{g} \rightarrow \mathfrak{g}$

P : parabolic subgroup of G ; i.e., the complex analytic subgroup corresponding to p .

The following theorem summarizes the major classical results which we will need. Details and proofs may be found in [7], [17].

Theorem 1.1 *Let the notation be as above. Then*

- (1) *If $b_2(M_0) = 1$ then M_0 is irreducible, G_0 is a compact centerless simple Lie group and K is a maximal compact proper subgroup of G_0 and G_c and has a 1-dimensional center.*
- (2) *P is a maximal parabolic subgroup of G , G_c acts transitively on $M = G/P$ and $G_c \cap P = K$.*
- (3) *$M_0 = G_0/K$ embeds holomorphically in M as an open G_0 -orbit, and M is a compact H.S.S.*
- (4) *Assume that $b_2(M) = 1$. Let $\Delta \subset \Phi$ be a base for the root system Φ . Then there is an $\alpha \in \Delta$ such that*

$$P = P(\Delta - \{\alpha\}) \quad (\text{cf., [10; Section 1]})$$

Moreover, the homogeneous line bundle L_α on G/P corresponding to α is ample. Thus G/P is projective algebraic.

Let $Y \subset M$ be a submanifold containing p_0 . Then Y is totally geodesic in M if and only if $T_{p_0}Y = n$ satisfies $[[n, n], n] \subset n$ – i.e., n is a Lie triple system. In this case $n + [n, n]$ is a Lie subalgebra of \mathfrak{g}_c . If H is the corresponding analytic subgroup of G_c then H acts transitively on Y and Y is again a symmetric space. Note: Y need not be Hermitian symmetric.

Lemma 1.2 *Let X be a simply connected symmetric space and $Y \subset X$ a totally geodesic simply connected submanifold of X . Assume that the connected group of isometries of Y , $I_0(Y)$, is semisimple. Then every isometry of Y extends to an isometry of X .*

Proof. Fix $y \in Y$ and let $n = T_{y,Y}$ be the corresponding Lie triple system. Let $\mathfrak{a} = n + [n, n]$ be the corresponding subalgebra of \mathfrak{g}_X – the Lie algebra of $I_0(X)$ where $I_0(X)$ is the connected component of the isometry group of X . Let $A \subset I_0(X)$ be the analytic subgroup corresponding to \mathfrak{a} and $K_A = A \cap K$, K the stability group in $I_0(X)$ of y . Then by [7, p. 225] $Y = A/K_A$. In particular, there is a natural mapping $\beta : A \rightarrow I(Y)$ and by [7, Remark 2, p. 211] the image is a closed analytic subgroup of $I(Y)$.

Let K' be the stability group of y in $I(Y)$. If $\gamma \in I(Y)$ and $\gamma \cdot y = y'$, then since A acts transitively on Y , there is an $h \in A$ which $hy' = y$. Thus $\beta(h)\gamma \in K'$. Since $I_0(Y)$ is semisimple, the Lie algebra k' of K' is spanned by the set

$$\{R_0(u, v) : u, v \in T_{y,Y}\}$$

by [9; Vol. II, XI 3.2(5)]. But for u, v in $T_{y,Y}$ we can write $u = d\beta(\bar{u}), v = d\beta(\bar{v})$ with \bar{u}, \bar{v} in \mathfrak{a} . Since on X , $R_y(u, v)(w) = -[[u, v], w]$ and Y is totally geodesic, we have

$$\begin{aligned} R_0(u, v)(w) &= -[[u, v], w] \\ &= -[[d\beta(\bar{u}), d\beta(\bar{v})], w] \end{aligned}$$

It follows that $d\beta[\mathfrak{a}, \mathfrak{a}] = k'$. Hence

$$d\beta : \mathfrak{a} \rightarrow \text{Lie}(I_0(Y))$$

is surjective. Since the image of β is closed, $\beta(A) = I_0(Y)$.

Now $I_0(Y)$ is simply connected so there exists a subgroup $A_0 \subseteq A$ such that $\beta : A_0 \rightarrow I_0(Y)$ is an isomorphism and $I_0(Y) \subseteq I(X)$. This proves the lemma.

Corollary 1.3 *Let Y be a totally geodesic Hermitian symmetric subspace of the H.S.S. M . Assume that Y and M are compact and that Y is irreducible. Let $G' = A(Y)$ and $G = A(M)$ denote the respective linear groups of holomorphic transformations of Y and M . Then there exists an isogeny $\tilde{G}' \rightarrow G'$ and a homomorphism $\tilde{G}' \rightarrow G$ such that*

$$Y = \tilde{G}'/\tilde{P}' \rightarrow M = G/P$$

is the inclusion map.

Proof. Since the connected groups of holomorphic isometries $I_0(Y)$ and $I_0(M)$ are dense in the connected components of G' and G (cf., [7, p. 211]) respectively, the corollary follows readily from the lemma.

Remark. Note that if Y is irreducible and $\mathfrak{g} = k + m$ is the Cartan decomposition of the Lie algebra of $I_0(Y)$ then m generates \mathfrak{g} , i.e., $\mathfrak{g} = m + [m, m]$.

We give now an alternative description of Hermitian symmetric spaces in terms of C -spaces. Let G be a connected complex simply connected Lie group with simple Lie algebra \mathfrak{g} of rank ℓ . Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let Φ denote the set of roots of \mathfrak{g} so

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Fix a base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ for Φ and let Φ^+ denote the set of positive roots. Fix a simple root α_r , $1 \leq r \leq \ell$ and put

$$\begin{aligned} \Phi_1 &= \{\alpha \in \Phi : n_r(\alpha) = 0\} \\ \Phi(n^+) &= \{\alpha \in \Phi^+ : n_r(\alpha) > 0\} \\ \Phi(n) &= \Phi_1 \cup \Phi(n^+) \end{aligned}$$

Define subalgebras of \mathfrak{g} by

$$\begin{aligned} \mathfrak{g}_1 &= \mathfrak{h} + \sum_{\alpha \in \Phi_1} \mathfrak{g}_\alpha \\ n^+ &= \sum_{\alpha \in \Phi(n^+)} \mathfrak{g}_\alpha \\ n &= \mathfrak{h} + \sum_{\alpha \in \Phi(n)} \mathfrak{g}_\alpha \end{aligned}$$

Then \mathfrak{g}_1 is reductive, n^+ is nilpotent and $n = \mathfrak{g}_1 + n^+$. Let \mathfrak{g}_c be a compact real form of \mathfrak{g} with $\mathfrak{g}_\mathbb{R} \cap \sqrt{-1}\mathfrak{h}_\mathbb{R} \neq (0)$. Let P be the complex analytic subgroup of G such that $\text{Lie } P = n$ and G_c the real analytic subgroup with $\text{Lie } G_c = \mathfrak{g}_c$. Put $K = G_c \cap P$. Then G_c acts transitively on G/P and induces the structure of a compact complex manifold on the homogenous space $G_c/K \simeq G/P$. The irreducible compact Hermitian symmetric spaces are given as follows:

(A III) Grassmannians : $Gr(r, \ell + 1) = (A_\ell, \alpha_r)$

(BDI) Quadric Hypersurfaces:

$$Q^N = \begin{cases} (B_\ell, \alpha_1) & \text{if } N = 2\ell - 1 \\ (D_\ell, \alpha_1) & \text{if } N = 2\ell - 2 \end{cases}$$

(CI) (C_ℓ, α_ℓ)

D III (D_ℓ, α_ℓ)

E III (E_6, α_1)

E VII (E_7, α_7)

Note that the group P defined above is a maximal parabolic subgroup – the standard parabolic defined by the subset $\Delta_r = \Delta - \{\alpha_r\}$ of Δ . Moreover, K is a maximal compact subgroup of G_c and the Levi-factor L of P is the centralizer in G of $T_r = \text{Ker } \alpha_r$. The Lie algebra of L is $k + ik$, $k = \text{Lie } K$.

A somewhat more concrete description of the classical compact H.S.S. is as follows: (cf., [17, p. 321].

(A III) Grassmannians: $G_c = SU(\ell + 1)$, $K = SU(\ell + 1) \cap \{U(r) \times U(\ell + 1 - r)\}$, G_c/K is the space of r -dimensional subspaces of $\mathbf{C}^{\ell+1}$

(BDI) The nonsingular quadratic hypersurface in \mathbf{P}^{N+1} defined by

$$z_0^2 + \cdots + z_{N+1}^2 = 0$$

$$G_c = SO(N + 2), K = SO(N) \times SO(2)$$

(CI) $Sp(n)/U(n) \subset G(n, 2n)$

The space consists of n -dimensional linear subspaces of \mathbf{C}^{2n} annihilated by a nondegenerate skew-symmetric bilinear form.

(D III) The subvariety $G_c/K = SO(2n)/U(n)$ of the same Grassmannian $G(n, 2n)$ consisting of n -dimensional subspaces annihilated by a nondegenerate symmetric bilinear form.

In the sequel we sometimes want to distinguish between $Gr(d, n)$ thought of as d -dimensional linear subspaces of $\mathbf{P}_{\mathbf{C}}^n$ and as $d + 1$ dimensional subspaces of \mathbf{C}^{n+1} . In those instances we use the notation $G(s, V)$ or $G(s, n)$ for the Grassmannian thought of as s -dimensional subspaces of the n -dimensional vector space V .

2 Minimal and Linear Embeddings of $\mathbf{P}_{\mathbb{C}}^r$ in Complex Grassmannians.

Let V be an $n + 1$ dimensional complex vector space and d a positive integer with $n \leq 2d < 2n$. Let $Gr(d, n)$ denote the Grassmann manifold of $d + 1$ dimensional subspaces of V or equivalently d -dimensional linear subspaces of $\mathbf{P}(V) \simeq \mathbf{P}_{\mathbb{C}}^n$. The manifold $Gr(d, n)$ is Hermitian symmetric of type A III. If Y is a compact complex submanifold of $Gr(d, n)$ which has everywhere positive holomorphic bisectional curvature then by [12] or [14] Y is biholomorphic to a complex projective space. If Y is also totally geodesic in $Gr(d, n)$ then Y is a symmetric subspace of $Gr(d, n)$. In this section we study such submanifolds Y which have minimal degree in $Gr(d, n)$.

Let $E(d, n)$ denote the universal subbundle over $Gr(d, n)$ and $Q(d, n)$ the universal quotient bundle. Then

$$0 \rightarrow E(d, n) \rightarrow Gr(d, n) \times V \rightarrow Q(d, n) \rightarrow 0$$

is exact and the fiber over $p \in Gr(d, n)$ of $E(d, n) \rightarrow Gr(d, n)$ is the subspace of V represented by the point p . For a holomorphic embedding $\varphi : \mathbf{P}_{\mathbb{C}}^r \rightarrow Gr(d, n)$ we define the degree of φ to be the degree of the line bundle $\det(\varphi^* E^v(d, n))$ where $E^v(d, n)$ is the dual of $E(d, n)$. As usual for a sequence of integers $0 \leq a_0 < a_1 < \dots < a_d \leq n$ we define the Schubert variety $\Omega(a_0, a_1, \dots, a_d)$ as follows: Fix a sequence of subspaces

$$V_0 \subset V_1 \subset \dots \subset V_d \subset P(V)$$

with $\dim V_r = a_r$. Then

$$\Omega(a_0, \dots, a_d) = \{L \in Gr(d, n) \mid \dim(L \cap V_i) \geq i\}$$

The classes of these cycles in $H_*(Gr(d, n), \mathbf{Z})$ generate the homology. Suppose $a_i = i + 1$ for $0 \leq i \leq d$. Then the Schubert variety $\Omega(1, 2, \dots, d, d + 1)$ is just the set of subspaces $L \subset V$ which are contained in V_d . Hence $\Omega(1, 2, \dots, d + 1) = Gr(d, V_d) = \mathbf{P}^{d+1}$.

The Plücker embedding of $Gr(d, n)$ is determined by the line bundle $\det(Q(d, n)) = \det(E^V(d, n))$. For any Schubert variety $\Omega(a_0, \dots, a_d)$ of dimension k we have (cf., [6; p. 274])

$$\deg \Omega(a_0, \dots, a_d) = \frac{k!}{a_0! a_1! \dots a_d!} \cdot \prod_{i < j} (a_j - a_i)$$

where $k = \sum_{i=0}^d a_i - \binom{d}{2}$. In particular for $\Omega(1, 2, \dots, d + 1)$, $k = d + 1$ and $\deg \Omega(1, 2, \dots, d + 1) = 1$.

Now let $\varphi : \mathbf{P}^r \rightarrow Gr(d, n)$ be a holomorphic embedding. We say that φ is minimal if $\varphi^*c_1(Q(d, n)) = [O_{\mathbb{P}^r}(1)]$ and that φ is linear if φ factors as $\varphi_0 \circ \varphi_1$ where $\varphi_1 : \mathbf{P}^r \rightarrow \mathbf{P}^{d+1}$ is a linear embedding of \mathbf{P}^r as a subspace of \mathbf{P}^{d+1} and φ_0 is an isomorphism \mathbf{P}^{d+1} with a Schubert variety $\Omega(1, 2, \dots, d+1)$ in $Gr(d, n)$. The embedding φ is totally geodesic if $\varphi(\mathbf{P}^r)$ is a totally geodesic submanifold of $Gr(d, n)$.

Lemma 2.1 *Let $\varphi : \mathbf{P}^1 \rightarrow Gr(d, n)$ be an embedding. Then φ is minimal if and only if it is linear.*

Proof. A linear embedding is clearly minimal. Conversely, if φ is minimal then φ^*Q is a rank $n - d$ vector bundle generated by its global sections. By [13; p. 22],

$$\varphi^*Q = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(e_{n-d})$$

with $e_1 \geq e_2 \geq \dots \geq e_{n-d} \geq 0$. Since $\det \varphi^*Q = \mathcal{O}_{\mathbb{P}^1}(1)$ we have $e_1 = 1$ and $e_i = 0$ for $i > 1$. Now consider the pull back of the universal sequence

$$0 \rightarrow E(d, n) \rightarrow (Gr(d, n) \times \mathbf{C}^{n+1}) \xrightarrow{\pi} Q(d, n) \rightarrow 0$$

to \mathbf{P}^1 . We have

$$0 \rightarrow E(d, n)|_{\mathbb{P}^1} \rightarrow \mathbf{C}^{n+1} \times \mathbf{P}^1 \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{n-d-1} \rightarrow 0$$

Let $\{s_0, \dots, s_n\}$ be a global frame for the trivial bundle $\mathbf{C}^{n+1} \times Gr(d, n)$ such that $\pi(s_{d+2}, \dots, \pi(s_n))$ span the trivial \mathcal{O}^{n-d-1} of φ^*Q over \mathbf{P}^1 . For any point $x \in \mathbf{P}^1$ the corresponding subspace $L_x \subset \mathbf{C}^{n+1} \times \mathbf{P}^1$ must lie in the subspace B spanned by $\langle s_0, \dots, s_{d+1} \rangle$. Thus $\varphi(\mathbf{P}^1) \subset Gr(d, B)$ hence P is linear. \square

For a line \mathcal{L} in $Gr(d, d+1) \simeq \mathbf{P}^{d+1}$ we have a simple description of \mathcal{L} as a Schubert variety. Let $B = C^{d+2}$. There is a d -dimensional subspace B_{d-1} of B such that

$$\begin{aligned} \mathcal{L} &= \{L \subset B \mid B_{d-1} \subset L\} \\ &= \Omega(0, 1, 2, \dots, d, d+2) \end{aligned}$$

In particular, if L_1, L_2 are in \mathcal{L} then

$$\dim(L_1 + L_2) \leq d+1 + d+1 - d = d+2$$

So if $L_1 \neq L_2$, $L_1 + L_2 = B$

Lemma 2.2 *Let $\varphi : \mathbf{P}^2 \rightarrow Gr(d, n)$ be a minimal embedding. Then φ is linear.*

Proof. Let $\mathcal{L}, \mathcal{L}'$ denote two distinct lines in \mathbf{P}^2 . Then there exists subspaces B and B' of \mathbf{C}^{n+1} of dimension $d+2$ and subspaces $B_{d-1} \subset B$, $B'_{d-1} \subset B'$ of dimension d such that

$$\begin{aligned}\varphi(\mathcal{L}) &= \{L \mid B_{d-1} \subset L \subset B\} \\ \varphi(\mathcal{L}') &= \{L' \mid B'_{d-1} \subset L' \subset B'\}\end{aligned}$$

I claim $B = B'$. Suppose not. Then since $\mathcal{L} \cap \mathcal{L}' \neq \emptyset$ there is a common subspace L_0 of B and B' of dimension $d+1$. Thus $\dim(B + B') = d+2 + d+2 - (d+1) = d+3$ and $\varphi(\mathcal{L}), \varphi(\mathcal{L}')$ both lie in $G(d+1, B'') = Gr(d, P(B''))$ where $B'' = B + B'$. Since B_{d-1} and B'_{d-1} are both subspaces of L_0 we can find a basis $\{e_0, \dots, e_{d+2}\}$ of B'' such that the following conditions are fulfilled

$$\begin{aligned}B_{d-1} &= \langle e_0, \dots, e_{d-1} \rangle \\ B'_{d-1} &= \langle e_1, \dots, e_d \rangle \\ L_0 &= \langle e_0, \dots, e_d \rangle \\ B &= L_0 + \langle e_{d+1} \rangle \\ B' &= L_0 + \langle e_{d+2} \rangle\end{aligned}$$

Then the subspace $L_1 = \langle e_0, \dots, e_{d-1}, e_{d+1} \rangle$ is in \mathcal{L} and $L_2 = \langle e_1, \dots, e_{d+2} \rangle$ is in \mathcal{L}' . Hence

$$\dim(L_1 + L_2) = d+3$$

But L_1 and L_2 correspond to points in $\varphi(\mathbf{P}^2)$ so lie in a minimally embedded line \mathcal{L}'' . By Lemma 2.1 \mathcal{L}'' is linearly embedded so by the above remark

$$\dim(L_1 + L_2) = d+2$$

This contradiction leads to the desired conclusion $B = B'$ and hence $\varphi(\mathbf{P}^2) \subset Gr(d+1, B')$ so is linearly embedded. \square

Let us fix a basis e_0, \dots, e_n of V establishing a fixed isomorphism with \mathbf{C}^{n+1} . Let $B \subset V$ be the subspace spanned by e_0, \dots, e_{d+1} . Then without loss of generality one may assume that a linearly embedded \mathbf{P}^{d+1} has image $G(d+1, B)$. Restricting the universal sequence to \mathbf{P}^{d+1} gives

$$0 \rightarrow E(d, n)|_{\mathbb{P}^{d+1}} \rightarrow \mathbf{C}^{n+1} \times \mathbf{P}^{d+1} \rightarrow Q(d, n)|_{\mathbb{P}^{d+1}} \rightarrow 0$$

If $x \in \mathbf{P}^{d+1}$ then the corresponding subspace L_x lies in B so the $n-d-1$ vectors e_{d+2}, \dots, e_n thought of as global sections of $\mathbf{C}^{n+1} \times \mathbf{P}^{d+1}$ map onto global sections of $Q(d, n)|_{\mathbb{P}^{d+1}}$ free from relations. This implies that

$$Q|_{\mathbb{P}^{d+1}} = L \oplus (\mathbf{C}^{n-d-1} \times \mathbf{P}^{d+1})$$

where L is a line bundle. Thus since $c_1(\varphi^*Q(d, n)) = 1$, $\varphi^*Q(d, n) = \mathcal{O}_{\mathbb{P}^{d+1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{d+1}}^{(n-d-1)}$. Conversely, if $\varphi^*Q(d, n) = \mathcal{O}_{\mathbb{P}^{d+1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{d+1}}^{(n-d-1)}$ we can find e_{d+2}, \dots, e_n global sections of $\mathbf{C}^{n+1} \times$

$Gr(d, n)$ which map onto generators of the trivial factor of $Q(d, n)|_{\mathbb{P}^{d+1}}$. Extending these to a basis e_0, \dots, e_d for $\mathbf{C}^{n+1} \times Gr(d, n)$ yields $L_x \subset \langle e_0, \dots, e_{d+1} \rangle$ for each $x \in \mathbf{P}^{d+1}$.

Theorem 2.3 *Let $\varphi : \mathbf{P}^r \rightarrow Gr(d, n)$ be an embedding with $1 \leq r \leq d+1$. Then φ is minimal if and only if it is linear.*

Proof. It suffices by the above remark to show that $\varphi^*Q(d, n) = \mathcal{O}_{\mathbb{P}^{d+1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{d+1}}^{(n-d-1)}$. Now for $r = 1$ or 2 the Lemmas 2.1, 2.2 yield the result. If $2 < r < d+1$ we have for any line $\mathcal{L} \subset \mathbf{P}^r$, φ/\mathcal{L} is minimal so $\varphi^*Q|_{\mathcal{L}} = \mathcal{O}_{\mathcal{L}}(1) \oplus \mathcal{O}_{\mathcal{L}}^{(n-d-1)}$. It follows that $\varphi^*Q(d, n)$ is uniform - i.e., for each $\mathcal{L} \subset \mathbf{P}^r$ $\varphi^*Q/\mathcal{L} = \mathcal{O}_{\mathcal{L}}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathcal{L}}(a_{n-d})$ with a_i constant. By Lemma 2.2, $\varphi^*Q(d, n)$ splits on a $\mathbf{P}^2 \subset \mathbf{P}^r$ also. Then by a theorem of Horrocks [13, p. 42] $\varphi^*Q(d, n)$ splits over \mathbf{P}^r so φ is linear. \square

The above arguments actually yield a more general result.

Corollary 2.4 *Let Q be a vector bundle on $\mathbf{P}_{\mathbf{C}}^n$ of rank d and generated by $n+d$ global sections. If Q is uniform of type $(1, 0, \dots, 0)$ then Q splits on \mathbf{P}^n , i.e., $Q \simeq \mathcal{O}_{\mathbb{P}^n}^n(1) \oplus \mathcal{O}_{\mathbb{P}^n}^{(d-1)}$.*

Proof. By hypothesis there exist a surjection

$$\mathcal{O}_{\mathbb{P}^n}^{n+d} \xrightarrow{\varphi} Q \rightarrow 0$$

Let (*): $0 \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^n}^{n+d} \xrightarrow{\varphi} Q \rightarrow 0$ be the corresponding exact sequence of locally free sheaves.

Over $\mathbf{P}^1 \subset \mathbf{P}^n$ we obtain

$$0 \rightarrow E|_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}^{n+d} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^{d-1} \rightarrow 0.$$

It follows that the morphism $\psi : \mathbf{P}^n \rightarrow Gr(n, n+d)$ defined by (*) restricts to a linear embedding on each $\mathbf{P}^1 \subset \mathbf{P}^n$. The proof of Lemma 2.2 shows that ψ is then linear on each $\mathbf{P}^2 \subset \mathbf{P}^n$. By the remarks preceding Theorem 2.3 $Q = \psi^*Q(n, d+n)$ splits on \mathbf{P}^2 into $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}^{d-1}$. By the theorem of Horrocks [13, p. 42] Q splits over \mathbf{P}^n . \square

In addition, we can obtain the following restriction on the existence of maps from projective spaces into Grassmannians.

Corollary 2.5 *If $K > d+1$ then every holomorphic map φ from $\mathbf{P}_{\mathbf{C}}^K$ to $Gr(d, n)$ with $c_1(\varphi^*Q(d, n)) = 1$ is constant.*

Proof. Suppose $K > d+1$ and let $W \subset \mathbf{P}_{\mathbf{C}}^K$ be a linear subspace of dimension $d+2$. If φ is holomorphic map from \mathbf{P}^K to $Gr(d, n)$ with $c_1(\varphi^*Q(d, n)) = 1$ then for every hyperplane $W' \subseteq W$,

the restriction of φ to W' is a minimal hence linear embedding. Suppose W_1 and W_2 are two distinct hyperplanes in W and let L_i be a line in W_i such that $L_1 \cap L_2 \neq \emptyset$ and $L_1 \not\subset W_2, L_2 \not\subset W_1$. We can find $B_i \subset \mathbf{C}^{n+1}$, $i = 1, 2$ such that

$$\varphi(W_i) = G(d+1, B_i)$$

and $B'_i \subset B_i$ of dimension d such that

$$\varphi(L_i) = \{L \in G(d+1, B_i) | B'_i \subset L \subset B_i\}.$$

Just as in the proof of Lemma 2.2 we can conclude that $B_1 = B_2 = B$. It follows that $\varphi(W') \subset G(d+1, B) \subset Gr(d, n)$ for every hyperplane $W' \subset W$ so $\varphi(W) \subset Gr(d+1, B)$. But then $\dim W = d+2$ and the only morphism from $W \cong \mathbf{P}^{d+2}$ to $G(d+1, B) \simeq \mathbf{P}^{d+1}$ are known to be constant. \square

The interested reader may compare the above result with the results of Tango in [15] and [16].

3 Some Representation Theory.

Let X be a compact H.S.S. and Y a totally geodesic compact Hermitian symmetric subspace of X . If Y is isomorphic to $\mathbf{P}_{\mathbf{C}}^{\ell}$ as H.S.S., then we have seen in Corollary 1.3 that there is a homomorphism

$$\rho : SU(\ell+1, \mathbf{C}) \rightarrow G$$

where G is the group of analytic automorphisms of X . If X is a Grassmannian, say $Gr(d, n)$, then ρ yields a homomorphism which we again call ρ

$$\rho : G_Y = SL(\ell+1, \mathbf{C}) \rightarrow SL(n+1, \mathbf{C})$$

so ρ is a representation of G_Y on $V = \mathbf{C}^{n+1}$. We want to describe these representations when $2\ell \geq n$.

Let $\lambda_1, \dots, \lambda_{\ell}$ be the fundamental dominant weights for $SL(\ell+1, \mathbf{C})$. We fix the maximal torus $T \subset SL(\ell+1, \mathbf{C})$ to be the diagonal subgroup and use the standard description of the roots as in [8, p.64]. In particular, if $d = \text{diag}(d_1, \dots, d_n) \in T$ the base for the root system ($B =$ upper triangular matrices) is given by

$$\alpha_i(d) = d_i d_{i+1}^{-1} \quad 1 \leq i \leq n$$

and

$$\lambda_i = \frac{1}{\ell+1} [(\ell-i+1)\alpha_1 + \dots + (i-1)(\ell-i+1)\alpha_{i-1} + i(\ell-i+1)\alpha_i + i(\ell-i)\alpha_{i+1} + \dots + i\alpha_{\ell}]$$

for $1 \leq i \leq \ell$. A straightforward computation gives

$$\lambda_i(d) = d_1 d_2 \cdots d_i.$$

If $e_1, \dots, e_{\ell+1}$ are the standard basis elements for $V = C^{n+1}$ then $e_1 \wedge \cdots \wedge e_i \in \bigwedge^i V$ is the highest weight vector corresponding to λ_i . Hence

$$\deg(\lambda_i) = \binom{n+1}{i}$$

Recall that for a dominant weight $\lambda = m_1 \lambda_1 + \cdots + m_\ell \lambda_\ell$ Weyl's character formula (cf. [7, VII, 10.2, 11.2]) gives

$$\deg \lambda = \left\{ \prod_{\alpha \in \Phi^+} \langle \lambda + \delta, \alpha \rangle \right\} / \prod_{\alpha \in \Phi^+} \langle \delta, \alpha \rangle.$$

Following Humphries [ibid] we can use the co-root α^\vee instead of α . Then for α height r

$$(\lambda + \delta, \alpha^\vee) = m_i + \cdots + m_{i+r-1} + r$$

if $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+(r-1)}$ is a root of height r and

$$(*) \quad \deg \lambda = \left\{ \prod_{\alpha \in \Phi^+} (\lambda + \delta, \alpha^\vee) \right\} / \prod_{\alpha \in \Phi^+} (\delta, \alpha^\vee)$$

Proposition 3.1 *Let $G = SL(\ell+1, \mathbf{C})$ with $\ell \geq 4$ and let V be a G -module of dimension d . Assume that $\ell+1 < d < 2\ell$. Then V contains a trivial G -module of dimension $r \geq d - \ell - 1$ as a direct summand. If $\ell \geq 5$ and $\ell+1 < d < 2(\ell+1)$ then V contains a trivial summand of dimension $r \geq d - \ell - 1$.*

Proof. Let $W \subset V$ be a nontrivial irreducible submodule of V with highest weight $\lambda = m_1 \lambda_1 + \cdots + m_\ell \lambda_\ell$. I claim that $m_1 + \cdots + m_\ell = 1$. Since $(\lambda + \delta, \alpha^\vee) = m_i + \cdots + m_{i+r-1} + r$ if $\alpha = \alpha_i + \cdots + \alpha_{i+r-1}$ it follows that if $\lambda > \lambda'$ in the lexicographic ordering then $\deg \lambda > \deg \lambda'$. If $\lambda = \sum_{i=1}^{\ell} m_i \lambda_i$ with $m_i > 0$ for some index i with $2 \leq i \leq \ell-1$ then $\deg \lambda_i = \binom{\ell+1}{i}$. But $\binom{\ell+1}{i} > 2(\ell+1)$ for i in this range and $\ell \geq 4$. It follows that $m_i = 0$ if $2 \leq i \leq \ell-1$.

Consider now $\lambda = m_1 \lambda_1 + m_\ell \lambda_\ell$. Again $\deg D > \min(\deg m_1 \lambda_1, \deg m_\ell \lambda_\ell)$. If $m_1 \geq 2$ then

$$\deg(m_1 \lambda_1) = m_1 \cdot 1^{\ell-1} \cdot (m_1 + 2) 2^{\ell-3} \cdots (m_1 + \ell) / \Delta > m_1 \deg \lambda_1 \geq 2 \deg \lambda_1 = 2(\ell+1)$$

Similarly, if $m_\ell \geq 2$ then $\deg(m_\ell \lambda_\ell) \geq 2 \deg \lambda_\ell = 2(\ell+1)$. Thus the hypothesis $d \leq 2(\ell+1)$ implies $m_1 \leq 1$ and $m_\ell \leq 1$. If $\lambda = \lambda_1 + \lambda_2$ then

$$\begin{aligned} \deg D &= \frac{2 \cdot 2^{\ell-2} \cdot 3 \cdot 2^{\ell-3} \cdots (\ell+2)}{\Delta} \\ &> 2 \deg \lambda_1 = 2 \deg \lambda_2 = 2(\ell+1) \end{aligned}$$

Hence we get a contradiction unless (m_1, m_2) is $(1, 0)$ or $(0, 1)$. The proposition now follows. \square

Proposition 3.1 will be used in Section 5 to determine the projective rank of the Grassmann manifolds.

4 The Chen-Nagano Classification of H.S.S.

In [5] B. Y. Chen and T. Nagano give a method of classifying compact symmetric spaces in terms of certain totally geodesic submanifolds. In this section we summarize their method and in the next section apply it to the problem of determining the projective ranks of the compact irreducible Hermitian symmetric spaces.

Let M be a symmetric space, $o \in M$ the origin and s_0 the geodesic symmetry of M fixing the point o . A smooth closed geodesic through o , or circle c for short, has an antipodal point p with $s_o(p) = p$. Write $M = G/K$ where G is the group of isometries of M (the closure of the group generated by all the symmetries $s_q, q \in M$) and K is the isotropy group of o . Let $M_+(p)$ denote the orbit $K \cdot p$. Then $M_+(p)$ is a totally geodesic submanifold of M [5 2.1]. Let σ_p be the involution of G given by $\sigma_p(g) = s_p g s_p^{-1}$. If $\mathfrak{g} = \mathfrak{k} + \mathfrak{n}$ is the Cartan decomposition of the Lie algebra $\mathfrak{g} = \text{Lie } G$ with respect to σ_0 then σ_p leaves \mathfrak{k} stable (s_0 and s_p commute) and induces the decomposition

$$\mathfrak{n} = \mathfrak{n}_+ + \mathfrak{n}_-$$

of \mathfrak{n} into the $+1$ and -1 eigenspaces of σ_p . Now $[[\mathfrak{n}_-, \mathfrak{n}_-], \mathfrak{n}_-] \subseteq \mathfrak{n}_-$ so \mathfrak{n}_- forms a Lie triple system. Consider the connected subgroup G_- of G corresponding to the Lie subalgebra $\mathfrak{k}_+ + \mathfrak{n}_-$. Let $M_-(p) = G_- \cdot p$. Then $M_-(p)$ is totally geodesic in M and the tangent space to $M_-(p)$ at p is the normal space to $M_+(p)$ at p in M (see [5, 2.2]).

Given a pair of antipodal points (o, p) on a circle c in M we have the system $(o, p, M_+(p), M_-(p))$. The group G acts naturally on the set of all such systems. Denote by $P(M)$ the orbit set. If $f : B \rightarrow M$ is an isometric totally geodesic embedding then there arises a mapping $P(f) : P(B) \rightarrow P(M)$ induced by the mapping carrying $(o, p, B_+(p), B_-(p))$ into $(f(o), f(p), M_+(f(p)), M_-(f(p)))$. Moreover, $f(B_+(p)) \subset M_+(f(p))$ and $f(B_-(p)) \subseteq M_-(f(p))$. We say $P(f)$ is a pairwise totally geodesic immersion. The following theorem summarizes the results of Chen-Nagano which we shall use.

Theorem 4.1 [5, Section 5] *Let o be a point fixed by K in the compact symmetric space $M = G/K$. For $s \in G$ let M^s denote the fixed point set of s . Then*

- (1) $M^{s_0} - \{o\}$ is the set of antipodal points on circles through o .

(2) For each $p \in M^{s_0} = \{0\}$ there exists an involutive automorphism $ad(b)$, $b \in K \cap \exp(n)$ such that

- 2(i) $M_+(p) = K(p)$ is a covering space of K/K^b
- 2(ii) $M_-(p)$, the connected component of M^b containing p is locally isometric with G^b/K^b
- 2(iii) the tangent spaces to the totally geodesic submanifolds $M_+(p)$ and $M_-(p)$ are the orthogonal complements of each other in $T_p(M)$.

(3) The rank of $M_-(p)$ equals the rank M and if K is connected, the rank K^b equals the rank of K .

(4) If a totally geodesic submanifold B of M has the same rank as M then

- (4(i)) $P(f) : P(B) \rightarrow P(M)$ is surjection, f the inclusion
- (4(ii)) The Weyl group $W(B)$ of B is a subgroup of $W(M)$
- (4(iii)) If $W(B)$ is isomorphic to $W(M)$, then $P(f)$ is bijective

(5) M is globally determined by $P(M)$; i.e., The set of isomorphism classes of compact irreducible symmetric spaces is in one-to-one correspondence with the set of the corresponding $P(M)$.

The following table gives the pairs $M_+(p)$ and $M_-(p)$ for the compact irreducible Hermitian symmetric spaces. In this table $T, G^R(k, n-k)$, $G^C(k, n-k)$, and $G^H(k, n-k)$ denote respectively the circle, the unoriented real Grassmann manifold $SU(n)/S(0(k) \times 0(n-k))$, the complex Grassmann manifold $SU(n)/S(U(k) \times U(n-k))$, and the quaternion Grassmann manifold $Sp(n)/Sp(k) \times Sp(n-k)$. A determination is outlined in [5, p. 409].

M	M_+	M_-	Remark	$\#P(M)$
$AIII(n) : G^C(p, q)$ $p + q = n$	$G^C(h, p-h) \times G^C(h, q-h)$	$G^C(h, h) \times G^C(p-h, q-h)$	$0 < h \leq p \leq q$	p
$BDI : G^R(p, q)$	$G^R(h, p-h) \times G^R(h, q-h)$	$G^R(h, h) \times G^R(p-h, q-h)$	$0 > h \leq p \leq q$	p
$CI(n) : Sp(n)/U(n)$	$G(h, n-h)$	$CI(k) \times CI(n-k)$	$0 > k \leq n-1$	n
$DIII(n) : SO(2n)/U(n)$	$G^C(k, n-k)$	$DIH(k) \times DIH(n-k)$	$0 < k < n$	$[\frac{n}{2}]$
$EIII$	$DIII(5)$	$S^2 \times G^C(5, 1)$		2
$EVII$	$EIII$	$S^2 \times G^R(10, 2)$		2

Table 4.2

4.3 An Example Let $M = \mathbf{P}_{\mathbb{C}}^n$. Then $M = SU(n+1)/S[U(n) \times U(1)]$ and $\text{rank}(M) = 1$. All geodesics in M are circles, have the same length and are permuted transitively by $SU(n+1)$ [7,

VII, 10.2, 11.2]. To find the possible geodesic pairs in M it suffices to consider just one circle. It is well-known that any projective line $L \subset \mathbf{P}_{\mathbb{C}}^n$ (embedded linearly; i.e., of degree 1) is totally geodesic in M so we may take the circle c to lie in L .

Viewing M as the set of n -dimensional subspaces of \mathbf{C}^{n+1} , L is the Schubert variety

$$L = \{W \in Gr(n, n+1) : W \supset W_0, W_0 \text{ a fixed } n-1 \text{ plane}\}$$

Let e_0, \dots, e_n be an orthonormal basis for \mathbf{C}^{n+1} such that $W_0 = \langle e_0, \dots, e_{n-2} \rangle$. If $W \in L$ and v is an element of W then $v = w_0 + ae_{n-1} + be_n$ with $w_0 \in W_0$. The assignment $v \mapsto [a, b]$ where $[a, b]$ are homogeneous point coordinates in $\mathbf{P}_{\mathbb{C}}^1$ defines an isomorphism of L with $\mathbf{P}_{\mathbb{C}}^1$. We take our circle c to be defined by

$$t \mapsto \langle e_0, \dots, e_{n-2}, \cos(t/2) \cdot e_{n-1} + \sin(t/2) \cdot e_n \rangle$$

Let $g \in R = S[U(n) \times U(1)] \cap St_{SU(n+1)} \langle e_0, \dots, e_{n-2}, e_n \rangle$. Then since $W_0 = \langle e_0, \dots, e_{n-2}, e_n \rangle \cap \langle e_0, \dots, e_{n-2}, e_{n-1} \rangle$, $gW_0 = W_0$ so $R \subseteq H = St_{SU(n+1)}(W_0) \cap S[U(n) \times U(1)]$. Hence we have a natural surjective map

$$Kp = K/R \rightarrow K/H \cong \mathbf{P}_{\mathbb{C}}^{n-1}$$

where $K = S[U(n) \times U(1)]$. Since $K \cdot p$ is a connected totally geodesic subspace and is a proper subspace of M and since $\mathbf{P}_{\mathbb{C}}^{n-1}$ is simply connected this map is an isomorphism and $K \cdot p = \mathbf{P}_{\mathbb{C}}^{n-1}$. It follows that $(M_+(p), m_-(p)) = (\mathbf{P}_{\mathbb{C}}^{n-1}, S^2 \simeq \mathbf{P}_{\mathbb{C}}^1)$ is the unique totally geodesic pair for $\mathbf{P}_{\mathbb{C}}^n$.

5 Projective Ranks of Compact H.S.S.

We determine the projective ranks of the irreducible compact Hermitian symmetric spaces in this section. We use the classification of Cartan as given in Table 4.2.

5.1 Type AIII.

We will show by induction on n that $\mathbb{P}_{\mathbb{C}}^{d+1}$ embedded linearly in $Gr(d, n)$ is a maximal totally geodesic complex projective submanifold of $Gr(d, n)$. If $n = 1$ there is nothing to prove since $Gr(0, 1) \cong \mathbb{P}_{\mathbb{C}}^1$. Assume the result is known for all $Gr(d', n')$ with $n' < n$ where $n' \leq 2d'$ as usual. Let $\mathbb{P}_{\mathbb{C}}^K$ be a maximal totally geodesic complex projective submanifold in $Gr(d, n)$. Since $Gr(d, d+1) \simeq \mathbb{P}_{\mathbb{C}}^{d+1}$ is totally geodesic in $Gr(d, n)$ when embedded linearly, $K \geq d+1$. Suppose $K > d+1$. Using Corollary 1.3 we can find a nontrivial homomorphism

$$\rho : SL(K+1, \mathbb{C}) \rightarrow SL(n+1, \mathbb{C})$$

Since $2d \geq n$, $2K \geq n$. Thus \mathbb{C}^{n+1} must split as an $SL(K+1, \mathbb{C})$ -module by Proposition 3.1. Say $\mathbb{C}^{n+1} \simeq \mathbb{C}^{K+1} \oplus \mathbb{C}^{n-K}$. Let $0 \in Gr(d, n)$ correspond to the subspace $W \simeq \mathbb{C}^{d+1} \subseteq \mathbb{C}^{K+1}$. Since $K > d+1$, W has codimension at least 2 in \mathbb{C}^{K+1} . Hence if P is the stability group of W in $SL(n+1, \mathbb{C})$ and $Q = P \cap SL(K+1, \mathbb{C})$ then $SL(K+1, \mathbb{C})/Q \simeq Gr(d, K)$ is not a complex projective space because $K-d > 2$. This contradiction implies $K = d+1$. Thus the projective rank of $Gr(d, n)$ is $d+1$.

Note that the maximal complex projective spaces $\mathbb{P}_{\mathbb{C}}^{d+1}$ in $Gr(d, n)$ are linearly embedded, are permuted transitively by $SU(n+1, \mathbb{C})$ and are parametrized by the $d+2$ -dimensional subspaces of \mathbb{C}^{n+1} , i.e., by $Gr(d+1, n)$.

5.2 Type BDI.

The Hermitian symmetric space

$$Q_m = SO(m+2)/SO(m) \times SO(2)$$

is biholomorphic to the complex quadric in $\mathbb{P}_{\mathbb{C}}^{m+1}$

$$Q_m : z_0^2 + z_1^2 + \cdots + z_{m+1}^2 = 0$$

We have the following result from Chen-Nagano [4]:

Theorem 5.2.1 *Let M be a totally geodesic complete connected Riemannian submanifold of Q_m .*

Then

1. *The embedding of M in Q_m is unique up to an isometry of Q_m .*
2. *If M is maximal in Q_m then M is one of the following:*
 - (i) $Q_m - 1$
 - (ii) *a local Riemannian product of two spheres $S^p \times S^q$, $p+q=m$*
 - (iii) *If $m=2n$, $\mathbb{P}_{\mathbb{C}}^n$ as Riemannian manifold*
3. *If M is not maximal then either $M \subseteq Q_{m-1}$ or $M \subseteq \mathbb{P}^n(\mathbb{R})$ – the real projective space of dimension n if $m=2n$.* □

The canonical decomposition of the Lie algebra $so(m+2)$ is cf. [9, Vol. II, p. 278] $so(m+2) = \eta + \mathfrak{p}$ where

$$\eta = so(2) + so(m) = \left\{ \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & B \end{pmatrix} : B \in so(m) \right\}$$

and

$$p = \left\{ \begin{pmatrix} 0 & 0 & -t_\xi \\ 0 & 0 & -t_\eta \\ \xi & \eta & 0 \end{pmatrix} : \xi, \eta \text{ column vectors in } \mathbb{R}^m \right\}$$

The complex structure of p is given by $J(\xi, \eta) = (-\eta, \xi)$. An inner product on $p \times p$ is given by

$$g((\xi, \eta), (\xi', \eta')) = 4/c \{ \langle \xi, \xi' \rangle + \langle \eta, \eta' \rangle \}$$

where \langle, \rangle denotes the standard Euclidean product of \mathbb{R}^n and c is a positive scalar. Both g and J are invariant under the adjoint action of $K = SO(2) \times SO(m)$ represented as

$$K = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta & 0 \cdots 0 \\ \sin \theta & \cos \theta & 0 \cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & B \\ 0 & 0 & \end{bmatrix} : B \in SO(m) \right\}$$

If $R(\theta) \times B$ is a typical element of k as above and $(\xi, \eta) \in p = \mathbb{R}^n \times \mathbb{R}^m$ then

$$ad(\mathbb{R}(\theta) \times B)(\xi, \eta) = (B\xi, B\eta)R(-\theta)$$

where the right hand side is a matrix product. The complex structure is the natural one so that

$$(\xi, \eta) = \begin{pmatrix} \xi_1 & \eta_1 \\ \vdots & \vdots \\ \xi_n & \eta_m \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 + \sqrt{-1}\eta_1 \\ \vdots \\ \xi_n + \sqrt{-1}\eta_m \end{pmatrix} \in \mathbb{C}^n$$

Thus the action of $SO(m) \subset K$ is the natural action of $SO(m)$ viewed as a real subgroup of $U(m)$.

The expression for $ad(R(\theta) \times B)(\xi, \eta)$ may be read as

$$ad(R(\theta) \times B)(\xi, \eta) = (B \cdot \underline{Z})e^{-i\theta}$$

where \underline{Z} is the complex vector with entries $\xi_i + \sqrt{-1}\eta_i$.

Let $m \subset p$ be a Lie triple system which is J -invariant. Then $p = m \oplus m^\perp$ where the orthogonal splitting is relative to the standard Hermitian metric on p . Since g is the real part of this metric (up to scalar factor $4/c$) we have an exclusion of $SO(r), r = \dim m$, into $SO(m)$ and a Lie algebra homomorphism $so(r) \rightarrow so(m)$ compatible with adjoint actions on $m \subseteq p$. This includes an inclusion of symmetric pairs

$$(so(r+2), so(r) \times so(2)) \rightarrow (so(m+2), so(m) \times so(2))$$

and it follows readily that the totally geodesic submanifold determined by m is the complex quadric Q_r imbedded in Q_m in some position.

Proposition 5.2.2 *The only totally geodesic complex submanifolds of Q_m are the complex quadrics Q_r $r < m$ and the complex projective spaces $\mathbb{P}_{\mathbb{C}}^r$, $r \leq [\frac{m}{2}]$.*

Proof. Note that the assertion 2(iii) of 4.2 says that for $m = 2n$, $\mathbb{P}_{\mathbb{C}}^n$ is a totally geodesic submanifold. When $m = 2n$, let $\mathbb{R}^m = \mathbb{R}^{2n}$ have a complex structure J_0 . Relative to this complex structure $SU(n)$ embeds in $SO(m)$. Similarly for $\mathbb{R}^{m+2} = \mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$. Any n -dimensional complex subspace has a natural induced orientation as a $2n$ -dimensional real space. Thus $G(n, n+1)$ can be mapped to $SO(m+2)/SO(2) \times SO(m)$ as follows: Let e_1, \dots, e_{n+1} be the standard orthonormal base for \mathbb{C}^{n+1} and for $W \subset \mathbb{C}^{n+1}$ an n -plane write $\mathbb{C}^{n+1} = W \oplus W^\perp$ as orthogonal direct sum. Orient W so that W and $\langle e_1, \dots, e_n \rangle$ have the same orientation $-\langle e_1, \dots, e_n \rangle$ and W are in the same orbit under $SU(n+1)$. Then orient W^\perp so that $W \oplus W^\perp$ has the same orientation as e_1, \dots, e_{n+1} . Define the image of W to be the oriented 2-plane W^\perp in $SO(m+2)/SO(2) \times SO(m)$. The Lie algebra of $su(n+1)$ embeds in $so(m+2)$ by

$$A + iB \rightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

from this one can check that $G(n, n+1) \rightarrow SO(m+2)/SO(2) \times SO(m)$ is a complex embedding.

5.3 Type CI.

Let $M = Sp(n)/U(n)$ be a H.S.S. of type CI. Then M is the subset of $G(n, 2n)$ consisting of n -dimensional subspaces of \mathbb{C}^{2n} which are totally isotropic with respect to a skew symmetric bilinear form [17, p. 232]. We take the matrix of this form to be

$$J_n = \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix}$$

where J_n is the $n \times n$ -identity matrix.

Let $V_1 \subset \mathbb{C}^{2n}$ be the subspace defined by $z_1 = \dots = z_n = 0$ where z_1, \dots, z_{2n} are the usual coordinates of \mathbb{C}^n . Let $V_2 \subset \mathbb{C}^{2n}$ be the subspace defined by the equations $z_i + z_{n+i} = 0$ $1 \leq i \leq n$. Then V_1 and V_2 represent points in M and $V_1 \cap V_2 = (0)$. If we take the base point to be $p_0 = [V_1] \in M$ then $M = G_c/K$ where $G_c = Sp(n)/(\pm I_{2n})$ and $K = G_c \cap \{g \in Sp(n, \mathbb{C}) | gp_0 = p_0\}$. Then

$$K = \left\{ \pm \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} : A \in U(n) \right\}$$

It follows from this description that M is a totally geodesic submanifold of $G(n, 2n)$.

According to Table 4.2 we expect that the projective rank of M is $n-1$. To see this consider $L \in G(n-1, V_1)$. An easy calculation shows that $\dim(L^\perp \cap V_2) = 1$ and that $W = L \oplus (L^\perp \cap V_2)$

is totally isotropic in \mathbb{C}^{2n} . This induces a morphism $G(n-1, n, V_1) \rightarrow M$ exhibiting a map $\rho : \mathbb{P}_{\mathbb{C}}^{n-1} \rightarrow M$. In fact, since $\mathbb{C}^{2n} = V_1 \oplus V_2$ the above map is just the assignment $L \mapsto L + (L^\perp \cap V_2)$ for $L \subset V_1$ an $n-1$ plane. Let $g_{12} \in Sp(n)$ be the element such that $g_{12}(V_1) = V_2$. Then $SU(n) = SU(V_1)$ acts in $V_1 \oplus V_2$ by $u(v_1, v_2) = (uv_1, g_{12}ug_{12}^{-1}v_2)$. Under this action $SU(n)$ acts transitively on $\rho(G(n-1), V_1)$. Since ρ is 1-1 and $SU(n)$ equivariant, $\rho(G(n-1, V))$ is totally geodesic in M .

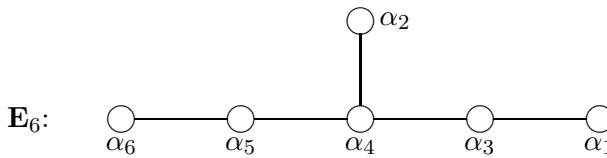
Thus the projective rank of $CI(n)$ is at least $n-1$. Since M is totally geodesic in $G(n, 2n)$ the maximal possible projective rank for M is $n = pr(G(n, 2n))$. But if $B \subset \mathbb{C}^{2n}$ is an $(n+1)$ -dimensional subspace then $G(n, B)$ cannot be contained in M . Indeed, let $L \subset B$ be completely singular of codimension one and suppose $B = L + \mathbb{C}v$. We have $L = L^\perp$ (this is easily seen for $L = V_1$ and since $Sp(n)$ acts transitively the same holds for L) choose codimension 1 subspaces L_1, \dots, L_n of L such that $L = L_1 + \dots + L_n$ and put $L'_i = L_i + \langle v \rangle \subseteq B$. If each L'_i is completely singular then $v \in L_1^\perp \cap L_2^\perp \cap \dots \cap L_n^\perp = (L_1 + \dots + L_n)^\perp = L^\perp$ a contradiction. Thus no $G(n, n+1)$ lies in M and $pr(M) = n-1$.

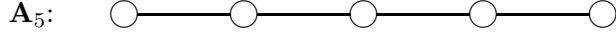
5.4 Type DIII

Let $M = SO(2n)/U(n)$ be the H.S.S. of type DIII(n). Then M may be identified with the submanifold of $G(n, 2n)$ consisting of completely singular n -dimensional subspaces with respect to the form $S_n = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$. Let e_1, \dots, e_{2n} be the standard basis for \mathbb{C}^{2n} and $V_1 = \langle e_1, \dots, e_n \rangle$, $V_2 = \langle e_{n+1}, \dots, e_{2n} \rangle$. Then V_1 and V_2 are completely singular. For $L \subset V_1$ a codimension one subspace, $\dim(L^\perp \cap V_2) = 1$ and $L + (L^\perp \cap V_2)$ is completely singular. Thus we have a map $G(n-1, V_1) \rightarrow M$ which one checks as in 5.3 is totally geodesic. Just as in 5.3, no $G(n, n+1)$ can lie in M which is itself totally geodesic in $G(n, 2n)$. Hence the projective rank of M is also $n-1$.

5.5 Type EIII

The exception space is $M = E_6/SO(10) \cdot SO(2)$. According to Table 4.2, M contains the unique symmetric pair $M_+ = DIII(5)$, $M_- = S^2 \times G^C(5, 1)$. We have seen that for any K the symmetric pair for $\mathbb{P}_{\mathbb{C}}^K$ is $(\mathbb{P}_{\mathbb{C}}^{K-1}, S^2)$. As the projective rank of DIII(5) is 4, by [5, p. 409] the pair $(\mathbb{P}_{\mathbb{C}}^4, S^2)$ is totally geodesically embeddable in M . It follows that the maximal possible value of $pr(M)$ is 5. From the Dynkin diagrams





We see that there exists a nontrivial homomorphism $sl(6, \mathbb{C})$ into E_6 . From [7, p. 507] the Cartan involution for M is induced by the permutation (1,6) (3,5) of the vertices of the Dynkin diagram for E_6 . Thus A_5 is stable under the Cartan involution. In particular, there exists a Lie triple system $m \subset E_6$ such that $m + [m, m] = sl(6, \mathbb{C})$. This together with the fact from Table 4.2 that $G(5, 1) \simeq \mathbb{P}_{\mathbb{C}}^5$ sits in M tells us that indeed the projective rank of EII is 5.

5.6 Type E VII

Let $M = E_7/E_6 \times SO(2)$ be the H.S.S. of type E VII. The unique maximal symmetric pair for M given by Table 4.2 is

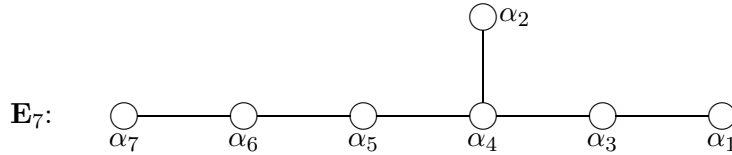
$$(M_+, M_-) = (EIII, S^2 \times G^{\mathbb{R}}(10, 2))$$

Since $pr(EIII) = 5$ we have a symmetric pair

$$(\mathbb{P}_{\mathbb{C}}^5, S^2)$$

contained in $E VII$ which suggest $pr(EVII) \leq 6$.

Again by considering the Dynkin diagram for E_7



we see that A_6 is in fact a subalgebra so again we have $sl(7, \mathbb{C}) \subset E_7$. Since $E_7 = k \oplus p$ with $k \simeq E_6 + \mathbb{R}$ we have $(sl(7, \mathbb{C}) \cap k)^c = sl(6, \mathbb{C}) + \mathbb{C}$. Thus the corresponding totally geodesic subspace is of type AIII and is $G(6, 7) \simeq \mathbb{P}^6$. Hence the projective rank of E_7 is 6. Summarizing we have the following result.

Theorem 5.6.1 *The projective ranks of the irreducible compact Hermitian symmetric spaces are as follows:*

$$(i) \quad pr[AIII(n, d)] = d, \quad n/2 \leq d < n$$

$$(ii) \quad pr[BDI(m)] = \lfloor \frac{m}{2} \rfloor$$

$$(iii) \quad pr[CI(n)] = n - 1$$

$$(iv) \quad pr[DIII(n)] = n - 1$$

$$(v) \quad pr[EIII] = 5$$

$$(vi) \quad pr[EVII] = 6.$$

6 Degree and Conjugacy Results.

Let $M = G/P$ be an irreducible Hermitian symmetric space with complex algebraic group of automorphisms G and stability group of $0 \in M$ the group P . As a C -space M has a canonical ample line bundle $O_M(1)$ defined as follows: If $P = L \cdot V$ is a Levi decomposition of P with V the unipotent radical of P , then L is the connected centralizer in P of a torus S . Let T be a maximal torus in G containing S such that the root system \sum is relative to T . If α is the root which defines the C -space, $S = (\ker \alpha)^0$ and hence α induces a character on L – the central character – which will again be called $\alpha : L \rightarrow \mathbb{C}^*$. Let $E_\alpha = G \times_P \mathbb{C}$ be the corresponding homogeneous line bundle on $M = G/P$. Then $O_M(1)$ is the sheaf of sections of E_α . It is known that $O_M(1)$ is very ample and we measure the degree of a subvariety $Y \subset M$ relative to the projective embedding defined by this line bundle.

Suppose $f : \mathbb{P}_{\mathbb{C}}^r \rightarrow M$ is a totally geodesic holomorphic embedding with image Y . To find the degree of f , or Y , it suffices to find the degree of the restriction of f to any projective line $L = \mathbb{P}_{\mathbb{C}}^1$ linearly embedded in $\mathbb{P}_{\mathbb{C}}^r$. With a view toward the conjugacy problem in mind we begin with the following well-known result:

Proposition 6.1 *There exists totally geodesic projective lines in M of degree one.*

Proof. We treat each type separately. If M is of type AIII the result follows from the fact that $G(d, d+1)$ has degree one in $Gr(d, n)$.

Suppose M is of type BDI. Viewing M as the complex quadric Q_n in $\mathbb{P}_{\mathbb{C}}^{n+1}$ we have a natural totally geodesic embedding of Q_2 in Q_n as

$$Q_2 = \{(Z_0, Z_1, Z_2, Z_3; 0, \dots, 0) | Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 = 0\}$$

Consider the holomorphic isomorphism

$$\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \rightarrow Q_2$$

given by

$$[u_0, u_1] \times [v_0, v_1] \xrightarrow{r} \left[\frac{u_0 v_0 + u_0 v_1}{2}, \frac{u_0 v_1 - u_0 v_0}{2i}, \frac{u_1 v_0 - u_1 v_1}{2}, \frac{u_1 v_0 + u_1 v_1}{2i} \right]$$

The image of the line $\mathbb{P}^1 \times [0, 1]$ is

$$L = \{[u_0/2; u_0/2i; u_0/2i; u_1/2; u_1/2i]\}$$

which clearly has degree one in the ambient projective space. By Mok [11, Section 1], L is totally geodesic in Q_2 hence the proposition holds for Q_n in general.

Now suppose M is of type *CIII* say $M = SP(n)/U(n)$. As in Section 5 we view M as n -dimensional completely singular subspaces of \mathbb{C}^{2n} . With V_1 and V_2 as in the last section we consider the mapping

$$\mathbb{P}^1 \rightarrow G(n, 2n)$$

given by $[u_0, u_1] \mapsto \langle e_1, \dots, e_{n-1}, u_0 e_n + u_1 e_{2n} \rangle$. It is easily verified that the image lies in M . Since the image also lies in $G(n, B)$ where $B = \langle e_1, \dots, e_n, e_{2n} \rangle$ it readily follows that the degree of the image is one and, by Mok [ibid] again, the image is totally geodesic in M . When M is of type *DIII* a completely analogous argument leads to the desired conclusion.

Finally, the exceptional cases *EIII* and *EVII* follow from the descriptions of the maximal totally geodesic complex projective spaces in these manifolds corresponding to the embeddings of $sl(6, \mathbb{C})$ and $sl(7, \mathbb{C})$ and the fact that the central characters in these cases restrict appropriately to $\mathcal{O}_{\mathbb{P}^r_{\mathbb{C}}}(1)$, $r = \text{projective rank of EIII (respectively EVIII)}$. \square

One might hope that the calculation of the degree of f could be accomplished using the above proposition. Unfortunately this is not the case. Moreover, the degree of a totally geodesic embedding $\mathbb{P}^s_{\mathbb{C}} \rightarrow M$ is not a constant function of s . The point is made with the following:

Example 6.2 There exists a totally geodesic holomorphic embedding of $\mathbb{P}^1_{\mathbb{C}}$ into Q_2 of degree 2.

Consider the subalgebra $\sigma \simeq so(3)$ of $so(4)$ consisting of matrices

$$\sigma = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \mid A \in so(3) \right\}$$

Then σ is stable under the cartan involution θ of Q_2 . If $m(\sigma) \subset n$ is the (-1) -eigenspace then

$$m(\sigma) = \left\{ \begin{bmatrix} 0 & 0 & -a & 0 \\ 0 & 0 & -b & 0 \\ a & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

so $m(\sigma)$ is also J -stable. The corresponding totally geodesic submanifold for this Lie triple system is therefore a complex submanifold. Consider the two elements X_1 and X_2 of σ ;

$$X_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $p = [1, i, 0, 0]$ as base point in Q_2

The geodesics given by $c_1 = \pi[\exp tX_1] \cdot p$ and $c_2 = \pi(\exp tX_2) \cdot p$ are

$$c_1 = \{[1, i \cos t, i \sin t, 0] \mid t \in \mathbb{R}\}$$

$$c_2 = \{[\cos t, i, \sin t, 0] \mid t \in \mathbb{R}\}$$

These curves lie on the algebraic curve

$$z_3 = z_0^2 + z_1^2 + z_2^2 = 0$$

in $\mathbb{P}_{\mathbb{C}}^3$ which is the image (up to linear automorphism in $\mathbb{P}_{\mathbb{C}}^3$) of the veronese map of degree 2 mapping $\mathbb{P}_{\mathbb{C}}^1$ into the plane $z_3 = 0$ in $\mathbb{P}_{\mathbb{C}}^3$. In particular, we have a totally geodesic complex projective line embedded in Q_2 having degree 2 and not 1. \square

We turn now to the spaces of type CI. Let V_1 and V_2 be the subspaces of \mathbb{C}^{2n} given in Section 5.3. We have a morphism

$$(6.3) \quad \rho : G(n-1, V_1) \times G(1, V_2) \rightarrow G(n, 2n) = G(n, V_1 + V_2)$$

given by $\rho(L, \ell) = L + \ell$. From the definition of the universal subbundle on $G(n, V_1 + V_2)$ it follows easily that $\rho^*(E(n, V_1 + V_2)) = p_1^*E(n-1, V_1) \oplus p_2^*E(1, V_2)$ where p_i denotes projection onto the i -th factor.

Consider the morphism $f : G(n-1, V_1) \rightarrow G(1, V_2)$ given by $f(L) = L^\perp \cap V_2$. Let Z be the graph of this mapping in $G(n-1, V_1) \times G(1, V_2)$. Then $\rho(Z)$ is precisely the totally geodesic submanifold of M given in 5.3. We want to compute the degree of the map $\tilde{f} = \rho \circ f_1$ where $f_1 : G(n-1, V_1) \simeq Z$. More precisely we compute the determinant of the bundle $[\tilde{f}^*E(n, V_1 + V_2)]^\vee$. Since $\rho^*E(n, V_1 + V_2) = p_1^*E(n-1, V_1) \oplus p_2^*E(1, V_2)$ it follows that $\tilde{f}^*E(n, V_1 + V_2) = f_1^*p_1^*E(n-1, V_1) \oplus f_1^*p_2^*E(1, V_2)$. Evidently, $f_1^*p_1^*E(n-1, V_1) = E(n-1, V_1)$. Now $E^\vee(1, V_2)$ is the line bundle corresponding to $\mathcal{O}(1)$ on $G(1, V_2) \simeq \mathbb{P}_{\mathbb{C}}^{n-1}$ and hence is generated by n -global sections. Thus $f_1^*p_2^*E(1, V_2)^\vee$ is a line bundle on $G(n-1, V_1) \simeq \mathbb{P}_{\mathbb{C}}^{n-1}$ generated by n -global sections so $f_1^*p_2^*E(1, V_2) \simeq \mathcal{O}(-1)$. It follows that $\tilde{f}^*E(n, V_1 + V_2) \simeq E(n-1, V_1) \oplus Q(n-1, V_1)^\vee$. Hence

$$\begin{aligned} \det[\tilde{f}^*E(n, V_1 + V_2)]^\vee &= \det[E(n-1, V_1)^\vee \oplus Q(n-1, V_1)] \\ &= \mathcal{O}(1) \oplus \mathcal{O}(1) \\ &= \mathcal{O}(2) \end{aligned}$$

An entirely analogous argument shows that if M is of type DIII, then the degree of the embedding given in 5.4 is also 2. Summarizing we have

Proposition 6.4. *Let M be an irreducible compact Hermitian symmetric space of type CI or DIII. Let $Y \subset M$ be a totally geodesic complex submanifold biholomorphic to $\mathbb{P}_{\mathbb{P}}^r, r = pr(M)$. Then $\deg(Y) \geq 2$.*

Proof. We have only to show that the case $\deg(Y) = 1$ cannot occur. If $\deg(Y) = 1$ then by Theorem 2.3 there is an $(n + 1)$ -dimensional subspace $B \subset \mathbb{C}^{2n}$ such that Y is a hyperplane in $G(n, B)$. Such a hyperplane consists of all n -dimensional subspaces of B containing a fixed line, say $\mathbb{C} \cdot v$. But v must be isotropic in this case. As we have noted in 5.3, B itself cannot be totally singular. Thus there exists $w \in B$ with $J(w, w) \neq 0$. The subspace spanned by v and w is contained in an n -dimensional subspace of B so there exists at least one such n -plane which is not totally isotropic. It follows that no hyperplane in $G(n, B)$ can be contained entirely in M and hence $\deg(Y)$ cannot be one. \square

Our next goal is to show that in fact the degree of Y is precisely 2 when M is of type CI or $DIII$. We will need a closer examination of totally geodesic complex projective subspaces of $G(n, 2n)$ of dimension $n - 1$. Let $Y \subset G(n, 2n)$ be such a submanifold. Then by 3.1 there exists a homomorphism $\rho : SU(n, \mathbb{C}) \rightarrow SU(2n, \mathbb{C})$ and a corresponding map $SL(n, \mathbb{C}) \rightarrow SL(2n, \mathbb{C})$. Let $W_0 \subset \mathbb{C}^{2n}$ be a fixed n -dimension subspace with orbit under $SU(n, \mathbb{C})$ isomorphic to $\mathbb{P}_{\mathbb{C}}^{n-1}$. Let H denote the isotropy group of W_0 in $SU(n, \mathbb{C})$. Then $SU(n, \mathbb{C})/H \simeq \mathbb{P}_{\mathbb{C}}^{n-1}$.

Lemma 6.5. *Let R be a compact subgroup of $SU(n, \mathbb{C})$ and suppose $SU(n, \mathbb{C})/R$ is biholomorphic to $\mathbb{P}_{\mathbb{C}}^{n-1}$. Then R is conjugate to $S[U(n - 1) \times U(1)]$ in $SU(n, \mathbb{C})$.*

Proof. Identify $\mathbb{P}_{\mathbb{C}}^{n-1}$ with $G(n - 1, n)$ so that $S[U(n - 1) \times U(1)]$ is the stability group of a point. Then $R \subset S[U(n - 1) \times U(1)]$. By [35, Theorem 6.1ii)] R is a maximal connected proper subgroup of $SU(n, \mathbb{C})$ so the desired equality follows. \square

The $SU(m, \mathbb{C})$ -module \mathbb{C}^{2n} decomposes as a direct sum $V_1 \oplus V_2$ with $\dim V_1 = n$ and $V_2 = V_1^\perp$. Since $H = S[U(n - 1) \times U(1)] \simeq SU(n - 1)$ is reductive each V_i decomposes as a sum of irreducible H -modules:

$$V_i = W_i \oplus L_i, \quad \dim W_i = n - 1, \quad \dim L_i = 1$$

Thus \mathbb{C}^{2n} decomposes as H -module

$$\mathbb{C}^{2n} = W_1 \oplus L_1 \oplus W_2 \oplus L_2$$

Now the submodule W_j is H -stable and not fixed by $SU(n, \mathbb{C})$ so $W_0 \simeq W_i \oplus L_j$ $i \neq j$ as H -modules. More precisely we have an element $\tau \in Hom_{H-mod}(W_i \oplus L_j, \mathbb{C}^{2n})$ with image W_0 .

Suppose for definiteness that $i = 1, j = 2$. Then

$$\begin{aligned}
& \text{Hom}_{H\text{-mod}}(W_i \oplus L_2, \mathbb{C}^{2n}) \\
&= \prod_{i=1,2} \text{Hom}_{H\text{-mod}}(W_1 \oplus L_2, W_i \oplus L_1) \\
&\simeq \prod_{i=1,2} [\text{Hom}_H(W_1, W_i) \times \text{Hom}_H(L_1, L_i)]
\end{aligned}$$

By Schur's lemma τ must be of the form $\tau(w, \ell) = (\beta w, 0, 0, \alpha, \ell)$ with $\alpha, \beta \in \mathbb{C}^*$. Thus $W_0 = W_1 \oplus L_2$ as H -module.

From this last equality we see that an element g in $SU(n, \mathbb{C})$ maps W_0 into $gW_1 \oplus gL_2$. It now follows that the orbit map $W_0 \rightarrow gW_0$ of $SU(n, \mathbb{C})$ into $G(n, 2n)$ is the same as the map described in 6.3 and hence $\deg(Y) = 2$ as desired. \square

We put the above calculations together and summarize the findings in the following results.

Theorem 6.6. *Let M be an irreducible Hermitian symmetric space with connected isometry group G_0 . Let Y be a totally geodesic complex projective subspace of $M, Y \simeq \mathbb{P}_{\mathbb{C}}^r$, with $r = \text{pr}(M)$. Let d be the degree of Y in M . Then*

- (i) $d = 1$ if M is of type AIII, EIII, or EVII.
- (ii) $d = 2$ if M is of CI or DIII.
- (iii) $d = 1$ or 2 if M is of type BDI.

Moreover, all such submanifolds Y of M of minimal degree are conjugate under G_0 .

Proof. The assertions in (i) and (ii) have already been established. As to (iii), by the discussion preceding 5.3 we see that each totally geodesic $\mathbb{P}_{\mathbb{C}}^1$ is contained in a complex quadric $Q_2 \simeq \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$. Now $Q_2 \subseteq \mathbb{P}_{\mathbb{C}}^3$ as the hypersurface $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$. Since $\mathbb{P}_{\mathbb{C}}^r$ is a complex submanifold of Q_2 , its embedding in $\mathbb{P}_{\mathbb{C}}^3$ must be by a complete linear system $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d))$. If $d > 2$, then the forms of degree d cannot satisfy the single quadratic relation above and yield an embedding so d is at most 2 and (iii) follows.

As for the conjugacy assertion, again in the case of type AIII this follows immediately. If M is of type BDI, then we need only consider $Q_2 \subseteq M$. Then the result follows from the fact that $SO(4, \mathbb{R})$ permutes the two-dimensional J -invariant Lie triple systems in T_0, Q_2 and clearly preserves the degree.

To treat the types CI and DIII we make use of the following:

Lemma 6.7. *Let M be a compact irreducible Hermitian symmetric space of type AIII, CI or DIII and $j : \mathbb{P}_{\mathbb{C}}^1 \rightarrow M$ a totally geodesic holomorphic isometric embedding of degree one. Then every geodesic circle in $j(\mathbb{P}_{\mathbb{C}}^1)$ has minimal length.*

Proof. Since CI and DIII are totally geodesic in AIII it suffices to prove the result for $M = G(d, n)$. For $G(d, n)$ the image of j is contained in $G(d, d+1)$ and a geodesic circle is given by (see example #3)

$$t \mapsto \langle e_0, \dots, e_{d-2}, (\cos t/2)e_{d-1} + (\sin t/2)e_d \rangle$$

$0 \leq t < 2\pi$. One checks easily that this has minimal length in $G(d, n)$ and the lemma follows. \square

By [7, VII, 11.2] the closed geodesics of minimal length in M are all conjugate. Now in the case of CI(n) or DIII(n) we can consider the geodesic (in the notation of 5.3)

$$t \mapsto \langle e_1, \dots, e_{n-1}, (\cos t/2)e_n + (\sin t/2)e_{2n} \rangle$$

The mid-point corresponds to the subspace $\langle e_1, \dots, e_{n-1}, ee_{2n} \rangle$ and the orbit under $SU(n, \mathbb{C})$ is precisely

$$\{W \oplus L | W \subset V_1, L \subset V_1^\perp\}$$

i.e., the image of the map in 6.3. Since the closed geodesics in M are conjugate so are the submanifolds which arise from 6.3, 6.4.

Finally, we discuss the exceptional types EIII and EVII. In each case according to Table 4.1 there exists a unique Hermitian pair (M_+, M_-) . Considering first the case EIII,

$$(M_+, M_-) = (DIII(5), S^2 \times G(1, 6))$$

If Y_1, Y_2 are the images of two totally geodesic embeddings of $\mathbb{P}_{\mathbb{C}}^5$ in M , then we may first assume $(Y_{1+}, Y_{1-}) = (Y_{2+}, Y_{2-})$ by the result for DIII. Then Y_1 and Y_2 meet along a hyperplane and have the same normal space in EIII so coincide.

If $M = EVII$, then

$$(M_+, M_-) = (EIII, S^2 \times G^{\mathbb{R}}(12, 2))$$

Again for Y_1, Y_2 images of totally geodesic maps from $\mathbb{P}_{\mathbb{C}}^6$ into M we may assume $(Y_{1+}, Y_{1-}) = (Y_{2+}, Y_{2-})$ so Y_1 and Y_2 meet along a $\mathbb{P}_{\mathbb{C}}^5$ and have the same normal space in EVII so coincide. This completes the proof.

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